A Mountaineering Accident: Not So Sticky After All

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The following problem appeared on a 1985 nationwide physics prize exam for university-level students, and was published with a solution[1] as an interesting question for discussion in a first-year mechanics course because it turns out to be far more complex than expected. It became even more intriguing for me when I realized that its creators had, within the constraints implied, overlooked a superior solution.

Here is the problem as it was originally presented:

> Some physics professors indulge in mountain climbing and have been known to take students along. You, the student, are standing on a 20-m wide horizontal ledge with loose gravel on it. The coefficient of friction μ (static and kinetic), between you and the ledge, is 0.80. You ask the professor to stand close to the edge while you take the perfect picture safely against the mountain wall. The professor slips and falls off the edge. You immediately grab onto the climbing rope which is tied onto the professor to brake his fall. The next ledge is 30 m below.
>
> You have many options to soften the fall of the professor, such as
>
> 1. You can hold tightly to the rope and be pulled over the edge too!
> 2. You can apply just so much braking force on the rope that you remain stationary while the professor slides toward the second ledge.
> 3. Hold tightly to the rope and let yourself be pulled forward toward the edge, but loosening your grip sufficiently before you reach the edge so as not to go over the edge yourself.
>
> Calculate the speed with which the professor hits the lower ledge for the strategy which damages the professor least and saves you from falling off the upper ledge.
>
> Neglect the mass of the rope, the friction of the rope on the edge, and assume you and the professor have the same mass (m).

F.L. Weichman's answer can be outlined as follows. In the first time "segment," the student applies as much resistive force to the rope as possible without accelerating himself (μmg). Then, after the professor has dropped 12.95 m, the student suddenly grabs very tightly onto the rope, causing an inelastic collision of sorts, absorbing some kinetic energy and halving the velocity of the professor. The student-professor system then accelerates together, reaching a maximum speed of 6.79 m/s just as the professor touches down. At this point, the student drops the rope and is decelerated steadily by the friction at his feet; he comes to a stop at the very edge of the ledge.

The oversight inherent in this solution is the result of presuming that the resistive force applied on the rope and professor by the student may not exceed the friction between student and gravel; in other words, the student, in this case, controls only the friction between his hands and the rope, causing this frictional force to be μmg in the first segment and an arbitrarily large value in the second.

He need not, however, limit his actions to changing the strength of his grip on the rope; the student can actively apply any retarding force on the rope (limited by his strength and endurance), even if it means pulling himself forwards along it. With this in mind, we can now make the braking force applied in the second segment a variable.

The other assumptions of Weichman's solution are sound. It is clearly preferable to "store" some kinetic energy for dissipation by the student after the professor has landed, and it is the solution that gives the student the largest velocity by the time the professor has traveled 30 m that will succeed in causing the least possible damage to the professor.

There will be no more than three segments of time with distinct forces and accelerations. By making the time periods variable, we solve a very general solution without presuming that any of the periods exist.

Segment 1. The student applies a backwards force of μmg to the rope for an unknown time t_1. The professor's acceleration during this period (since $ma = mg - \mu mg$: spatial values for each of the two individuals will be positive in the direction of motion) is thus $a_{1P} = (1 - \mu)\, g$, and the student's acceleration is 0.

Segment 2. The student pulls on the rope with a tension F. He will be able to continue doing this even if his velocity becomes greater than that of the rope and professor—it will be just as though he were pulling himself forward along a moving ski rope-tow. The student will continue stealing momentum for time t_2 until the professor lands. During the second segment, the professor's net acceleration, due to gravity and the student's pull, is $a_{2P} = g - F/m$; that of the student, due to friction and the rope's equal and opposite pull, is $a_{2S} = -\mu g + F/m$.

Segment 3. As the professor reaches the lower ledge, the student drops the rope and is on his own. In the remaining distance on the upper ledge, then, the student slows down with an acceleration $a_{3S} = -\mu g$ due to friction. This will take a time t_3.

With these parameters, we have only four restrictions on the motion of student and professor:

First, the professor must reach the bottom of the drop exactly at the end of the second segment: $\Delta d_1 + \Delta d_2 = 30$ m, and each $\Delta d = (v_i)\,t + \frac{1}{2}at^2$, so

$$(\tfrac{1}{2}a_{1P}t_1^2) + [(a_{1P}t_1)\,t_2 + \tfrac{1}{2}a_{2P}t_2^2] = 30\text{ m} \quad (1)$$

The best possible case for the professor's landing would be a zero velocity. With the extra braking efficiency in our present model, this can be easily achieved, and will become our second stipulation (if it was not possible, we would simply get no solution to our equations): $\Delta v_1 + \Delta v_2 = 0$, or

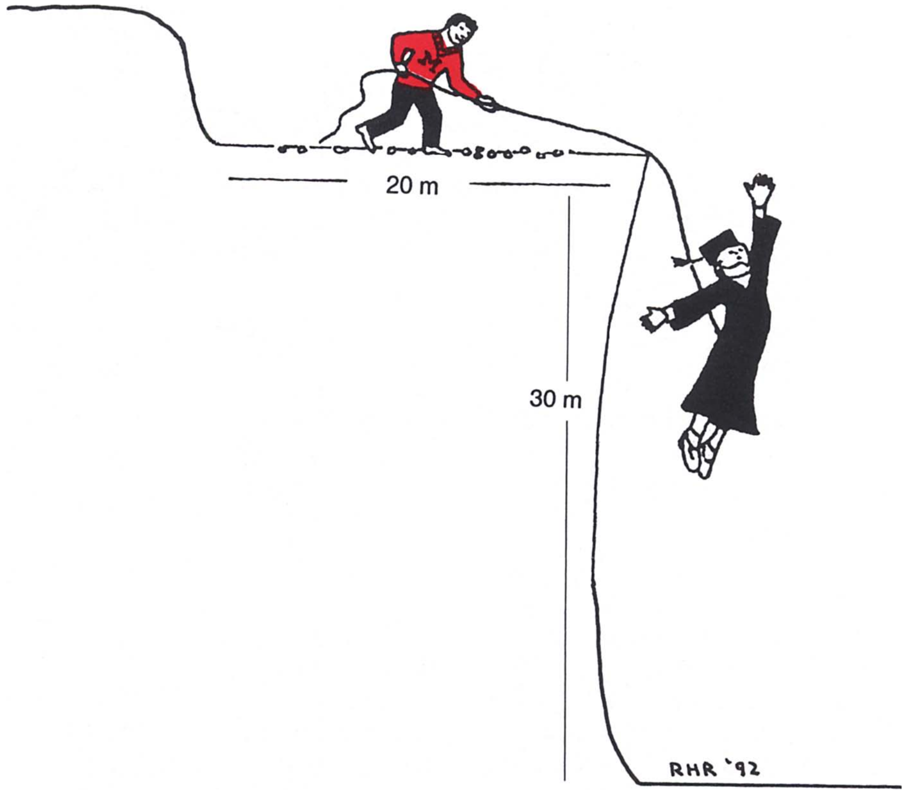
$$a_{1P}t_1 + a_{2P}t_2 = 0 \quad (2)$$

Similarly, the student must come to a rest, and preferably at the very brink of the upper ledge; through analogous logic, we get:

$$(\tfrac{1}{2}a_{2S}t_2^2) + [(a_{2S}t_2)\,t_3 + \tfrac{1}{2}a_{3S}t_3^2] = 20\text{ m} \quad (3)$$

$$a_{2S}t_2 + a_{3S}t_3 = 0 \quad (4)$$

Keeping the objective in mind—to minimize damage to the professor—our goal now becomes finding the values that result in imparting the least acceleration (g's) to the professor, as long as the assumption of a zero final velocity holds. Equation (3) is responsible for ensuring a minimum acceleration.



Substituting in the expressions for a_{1P}, a_{2S}, a_{2P}, and a_{3S}, and solving the system of four equations for F, we get a quadratic equation with a unique permissible solution of $F = 2\mu mg$; our assumption stands.

Solving for the other variables, we see that the two net accelerations undergone by the student are the same ($a_{2S} = -a_{3S}$), as are their durations ($t_2 = t_3$). Thus, in order to reach the maximum peak velocity with the lowest accelerations, the student travels exactly 10 m in Segment 1 and 10 m in Segment 2. With $g = 9.8$ m/s^2, the student's maximum velocity (at the end of Segment 2, as the professor lands) is 12.5 m/s, while in Weichman's solution it is only 6.79 m/s. Much more energy is converted to heat, too; in both cases there is 20 m of μmg friction between the student and gravel, but because the magnitudes of the accelerations in the second segment are greater in this solution, more time can be spent in the first segment: the student lets 22.49 m of rope slip through his hands with a frictional force of μmg, instead of 12.95 m.

There is a small amount of kinetic energy absorbed during the collision in Weichman's solution. The collision does not occur here, although it is contrary to the aim of reducing damage, anyway, since the energy is presumably absorbed by the pair's bodies in the sharp yank.

Finally, and most importantly, the maximum acceleration of the student a_{2S} is only 0.8 g, and the maximum acceleration of the professor a_{2P} is now only 0.6 g—quite a leisurely ride!

Note that each of the force and acceleration values is an average over its respective segment; although the student must obviously have an unusual aptitude for perception of periods of time, the force he applies during the second segment need not be strictly constant.

Further fun with this situation could include dropping the assumption of equal masses, or making the coefficient of friction vary with speed. As well as providing a good exercise in creative problem solving and analysis of linear dynamics, this question will undoubtedly help to reassure any apprehensive climbers about the sport's safety.

Reference

1. F.L. Weichman, *Phys. Teach.* **23**, 358 (1985).